The behaviour of a laminar compressible boundary layer near a point of zero skin-friction

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It is shown, with a high degree of certainty, that a general compressible laminar boundary layer can develop a singularity at a point of zero skin-friction only if the heat transfer at that point is zero.

1. Introduction

The irregular behaviour of an incompressible boundary layer near a point of zero skin-friction was first noticed by Howarth (1938) in a numerical computation. Subsequently the behaviour of the flow in this neighbourhood was investigated by Goldstein (1948) and Stewartson (1958); their work was improved and extended to include the effect of suction by Terrill (1960) who also gave an admirable account of the present position. These authors showed that a formal expansion of the stream function about the point of zero skin-friction can include non-integral powers of x, the distance upstream from this point, and eventually powers of $\log x$ whose coefficients are complicated functions of $y/x^{\frac{1}{4}}$, where y measures distance normal to the wall. Further numerical calculations, by Hartree (1939) and Leigh (1955) have confirmed and extended Howarth's results and it has been possible to join them up with the expansions about the point of zero skin-friction. Thus the existence of the singularity can be regarded as fully established and understood at the present time.

This property of the incompressible boundary-layer equations is reflected in the Falkner–Skan equations, a family of ordinary differential equations derived from them and characterized by a parameter β (Hartree 1937). When subject to appropriate boundary conditions, solutions of these equations only exist if $\beta \geqslant -0{\cdot}199$, and the skin-friction vanishes at the minimum β where, regarded as a function of β, it has an algebraic singularity.

The compressible boundary layer is of course much more complicated. However, if we assume that the Prandtl number σ is unity, that the viscosity μ is proportional to the temperature and that there is no heat transfer to or from the wall, the governing equations can be reduced to the incompressible form. Accordingly, in these circumstances the compressible boundary layer also develops a singularity at a point of zero skin-friction. Recently, however, two numerical integrations have been carried out with one of these restrictions, that of zero heat transfer from the wall, relaxed. Poots (1960) has considered the case of a constant wall temperature greater than its stagnation temperature so that there is heat transfer into the fluid. His solution showed no signs of irregularity near the point where the skin-friction vanishes. Curle (1958) has reported several cases in one of which the wall temperature was constant and less than the stagnation temperature. Great difficulties occurred in completing this solution right down to the point of zero skin-friction and it is possible, although not a certainty, that these arose because in fact the flow became singular *before* the skin-friction vanished.

Again, if the condition of zero heat transfer is relaxed to permit a constant wall temperature, then the compressible boundary-layer equations can be reduced to a form similar to that of the Falkner–Skan equation, although to be sure there is an additional equation for the temperature in the layer. The pair of equations have been integrated numerically in a variety of cases by Cohen & Reshotko (1956). From their work it appears that, if the temperature and velocity are subject to the appropriate boundary conditions, solutions can only be obtained if β is greater than a certain minimum value β_0 which is a function of the wall temperature. At $\beta = \beta_0$, the skin-friction is not zero unless the heat transfer from the wall is zero too, being positive if heat is transferred from the fluid to the wall and negative otherwise.

As a result of these findings it is of interest to look at the behaviour of a compressible boundary layer near a point of zero skin-friction to see whether it develops a singularity there. We shall find that for a general compressible fluid the boundary layer cannot have a singularity at the point of zero skin-friction. The chief and probably the only exception is if the heat transfer from the wall is zero there. No other exception could be found but the arguments, although strongly suggesting that there is only one, are not conclusive. The main assumptions of the paper are that the singularity, if it exists, is of the same character as that assumed by Goldstein (1948) and that the tangential stress in the fluid first vanishes at the wall. The argument is developed first for a model fluid in which the Prandtl number $\sigma = 1$ and the viscosity is proportional to the absolute temperature. It is then shown that the differences between this fluid and a real fluid are marginal, affecting the details of the solution but not its essential character.

The importance of these results is twofold. First it shows that computers of boundary layers must no longer expect the end point of their calculations to be where the skin-friction vanishes. It may occur earlier. Secondly, it indicates that experimenters may no longer be justified in regarding separation† as the point where the skin-friction vanishes. From a mathematical stand-point, at infinite Reynolds number the main stream leaves the wall when the boundary layer,

† Separation is used in the literature on boundary layers to refer to one or more of three phenomena, viz. (i) the point of the wall at which the tangential component of the stress in the boundary layer vanishes, i.e. the skin-friction is zero; (ii) the place at which the solution of the boundary-layer equations develop a singularity, so that the equations break down and cannot be continued further downstream; (iii) the place at which the main inviscid stream detaches itself from the wall. In an incompressible fluid no confusion arises because theory and experiment together indicate that these three points are identical, or very nearly so. However, the conclusions of this paper strongly suggest that these three points are not identical for a general compressible boundary layer. Accordingly, the use of the word separation is restricted here to its natural sense and refers only to the place at which the main stream detaches itself from the wall.

joining it to the wall, breaks down. If breakdown occurs before the skin-friction vanishes, which can no longer be ruled out, the main stream leaves the wall while the skin-friction is still positive. In this case an experiment, carried out at a finite but large value of the Reynolds number, would show a rapid thickening of the layer in a region of positive skin-friction. It has not been possible, however, to show that the boundary layer breaks down somewhere. Thus an analysis based on Goldstein's method (1930) indicated that a singularity with centre on the wall could not occur at a point of positive skin-friction. Presumably breakdown does occur, from the evidence available, and it is inferred that the centre of the singularity is no longer at the wall, being instead at an interior point of the boundary layer.

2. The model equations

To begin with we shall suppose that the fluid has a Prandtl number $\sigma = 1$ and viscosity proportional to the absolute temperature while the main stream outside the boundary layer is irrotational and homenergic. The equations governing the flow in the boundary layer may then be reduced to the form (Stewartson 1949)

$$\frac{\partial \psi}{\partial Y}\frac{\partial^2 \psi}{\partial X \partial Y} - \frac{\partial \psi}{\partial X}\frac{\partial^2 \psi}{\partial Y^2} = V\frac{dV}{\partial X}(1+S) + \nu_0 \frac{\partial^3 \psi}{\partial Y^3}, \tag{2.1}$$

$$\frac{\partial \psi}{\partial Y}\frac{\partial S}{\partial X} - \frac{\partial \psi}{\partial X}\frac{\partial S}{\partial Y} = \nu_0 \frac{\partial^2 S}{\partial Y^2}, \tag{2.2}$$

in which ψ is the stream function, S is simply related to the absolute temperature, X and Y correspond to distances measured along and perpendicular to the wall respectively, V is proportional to the Mach number of the main stream just outside the boundary layer and ν_0 is the kinematic viscosity at some standard place. The appropriate boundary conditions are that

$$\partial\psi/\partial Y \to V(X), \quad S \to 0 \quad \text{as} \quad Y \to \infty, \tag{2.3}$$

$$\psi = \partial\psi/\partial Y = 0 \quad \text{at} \quad Y = 0, \tag{2.4}$$

S and ψ are prescribed at some initial station of X and either

$$S = t(X) \quad \text{at} \quad Y = 0, \tag{2.5}$$

or

$$\partial S/\partial Y = 0 \quad \text{at} \quad Y = 0, \tag{2.6}$$

where $t(X)$ is a prescribed function of X, derived from a prescribed temperature distribution at the wall.

Without loss of generality we may suppose that the skin-friction vanishes at $X = Y = 0$, i.e. that

$$\frac{\partial^2 \psi}{\partial Y^2} = 0 \quad \text{when} \quad X = Y = 0, \tag{2.7}$$

and denote by the suffix 0 the value of a function at the origin. Then following Goldstein (1948) we introduce non-dimensional variables

$$\xi = \left(\frac{-X}{l}\right)^{\frac{1}{4}}, \quad \eta = R^{\frac{1}{2}}\left(\frac{l}{-4X}\right)^{\frac{1}{4}} Y, \quad \psi = 2^{\frac{1}{2}}\left(\frac{-X}{l}\right)^{\frac{3}{4}} f(\xi, \eta), \quad S = t_0 + (1+t_0)\, g(\xi, \eta), \tag{2.8}$$

where $l = -V_0/[V_0'(1+t_0)]$, $R = V_0 l/\nu_0$, and V' has been written for dV/dX.

Further we assume that, near $X = 0$,

$$-V\frac{dV}{dX} = -V_0\frac{dV_0}{dX}(1+P_1\xi^4+P_2\xi^8+\ldots), \tag{2.9}$$

and, in the case when S is prescribed at the wall,

$$g(\xi,0) = \xi^4 S_1+\xi^8 S_2+\ldots, \tag{2.10}$$

where the P's and S's are all known. Substituting (2.8) and (2.9) into (2.1) and (2.2), we find that f and g satisfy

$$\frac{\partial^3 f}{\partial\eta^3}-3f\frac{\partial^2 f}{\partial\eta^2}+2\left(\frac{\partial f}{\partial\eta}\right)^2+\xi\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\,\partial\eta}-\frac{\partial^2 f}{\partial\eta^2}\frac{\partial f}{\partial\xi}\right) = (1+g)\sum_{r=0}^{\infty}P_r\xi^{4r} \tag{2.11}$$

and

$$\frac{\partial^2 g}{\partial\eta^2}-3f\frac{\partial g}{\partial\eta}+\xi\left(\frac{\partial f}{\partial\eta}\frac{\partial g}{\partial\xi}-\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}\right) = 0. \tag{2.12}$$

Following the argument for incompressible flow, which is a special case of the present one, and in which $g \equiv 0$, we assume to begin with that f and g can be expanded in a series of integral powers of ξ, whose coefficients are functions of η. As in that case it will be found necessary in certain circumstances to modify the series by adding extra terms in which powers of ($\log\xi$) occur as multiplicative factors. In incompressible flow, once we accept the choice of variables in (2.8), it can be shown that there is no alternative to the form of the final expansion, but there *are* other possibilities different from (2.8) for the basic variables in the expansion. However, it can be shown that if the variables $x^{1/n}$, $y/x^{1/n}$, $n \neq 4$ are chosen, the pressure gradient is not a controlling feature of the flow near the point of zero skin-friction, and further, the solution is either self-contradictory or the stress vanishes in the interior of the fluid before it vanishes at the wall. It may be that the singularity is controlled by such variables in the present problem, but it does not seem likely because the effect on heat transfer only manifests itself through the pressure gradient, and further, there is no evidence from numerical calculations that the stress vanishes in the interior of the fluid before it vanishes at the wall. Another possibility is that the singularity may be of a different kind from that envisaged here and in fact of an entirely novel kind. The evidence from the numerical calculations is perhaps relevant here; while the work reported by Curle was equivocal, that of Poots was clear, and since it is in agreement with the general conclusions of this paper there seems little point in searching further.

Accordingly we write

$$f(\xi,\eta) = \sum_{n=0}^{\infty} f_n(\eta)\,\xi^n + \text{extra terms involving powers of } (\log\xi), \tag{2.13}$$

$$g(\xi,\eta) = \sum_{n=0}^{\infty} g_n(\eta)\,\xi^n + \text{extra terms involving powers of } (\log\xi). \tag{2.14}$$

It is convenient to include these terms in the f_n, g_n so that if at any stage it is necessary to include $\log\xi$ it will be done by assuming that f_n, g_n are polynomials in $\log\xi$.

The boundary conditions are that

$$f_n(0) = f_n'(0) = 0, \tag{2.15}$$

that either

$$g_{4r}(0) = S_r \quad \text{for} \quad r = 1, 2, \ldots \quad \text{and} \quad g_n(0) = 0 \quad \text{otherwise}, \tag{2.16}$$

for a prescribed wall temperature, or

$$g_0(0) = 0, \quad g_n'(0) = 0 \quad \text{for} \quad n \geqslant 1, \tag{2.17}$$

if the heat transfer at the wall is zero. Further f_n, g_n are not exponentially large at infinity, which follows from the assumption that the singularity, if it exists, is centred at the origin.

If we substitute (2.13) and (2.14) into (2.11) and (2.12), we find that f_0, g_0 satisfy

$$f_0''' - 3f_0 f_0'' + 2f_0'^2 = 1 + g_0, \quad g_0'' - 3f_0 g_0' = 0. \tag{2.18}$$

If, from these equations, f_0 is ultimately negative it means that f_0' must tend to a negative limit or to zero from below. In either case the stress must vanish in the fluid before it vanishes at the wall, which is excluded. However, if f_0 is ultimately positive and $g_0'(0) \neq 0$, $g_0' \to \infty$ exponentially which is also excluded. Accordingly, we must have $g_0 \equiv 0$, whence

$$f_0 = \tfrac{1}{6}\eta^3. \tag{2.19}$$

The equations for f_1, g_1 are

$$f_1''' - \tfrac{1}{2}\eta^3 f_1'' + \tfrac{5}{2}\eta^2 f_1' - 4\eta f_1 = g_1, \quad g_1'' - \tfrac{1}{2}\eta^3 g_1' + \tfrac{1}{2}\eta^2 g_1 = 0. \tag{2.20}$$

The homogeneous equation for f_1 and indeed the corresponding homogeneous equations for f_n have been extensively discussed by Goldstein (1948) and Terrill (1960). The general homogeneous equation for g_n is new and is discussed in Appendix A to this paper. It follows from this appendix that the only acceptable solution of the equation for g_1 is

$$g_1 = B_1 \eta, \tag{2.21}$$

where B_1 is an arbitrary constant at present, unless the heat transfer vanishes at the wall when it is zero. For the moment we shall assume it does not vanish. With the solution (2.21) for g_1, the solution of (2.20) becomes

$$f_1 = \alpha_1 \eta^2 + \tfrac{1}{24} B_1 \eta^4, \tag{2.22}$$

where α_n is an arbitrary constant of the equation for f_n equal to $\frac{1}{2}f_n''(0)$. Similarly, the equation for g_2 is

$$g_2'' - \tfrac{1}{2}\eta^3 g_2' + \eta^2 g_2 = 4f_1 g_1' - f_1' g_1 = 2\alpha_1 B_1 \eta^2, \tag{2.23}$$

of which the solution is

$$g_2 = 2\alpha_1 B_1 (1 - G_2), \tag{2.24}$$

where $G_2(\eta)$ is the complementary function of (2.23) which is algebraic at infinity and such that $G_2(0) = 1$. Equation (2.24) is the appropriate solution for a prescribed temperature at the wall. If the heat transfer from the wall is zero, $B_1 = 0$ and the appropriate solution is $g_2 = 0$. The equation for f_2 is

$$f_2''' - \tfrac{1}{2}\eta^3 f_2'' + 3\eta^2 f_2' - 5\eta f_2 = -4\alpha_1^2 \eta^2 + \tfrac{1}{3}\alpha_1 B_1 \eta^4 + g_2(\eta). \tag{2.25}$$

The complementary functions of (2.25) are $\eta^2, \eta+\eta^5/30$ and a function which is exponentially large at infinity. Hence (2.25) has a solution with a double zero at the origin and algebraic at infinity if and only if

$$\int_0^\infty \left(\eta^2 - \frac{\eta^6}{10}\right)\{g_2(\eta) + \tfrac{1}{3}\alpha_1 B_1 \eta^4 - 4\alpha_1^2\eta^2\}\, e^{-\frac{1}{8}\eta^4}\, d\eta = 0, \tag{2.26}$$

i.e. if

$$\alpha_1 B_1 \int_0^\infty \left(\eta^2 - \frac{\eta^6}{10}\right) G_2(\eta)\, e^{-\frac{1}{8}\eta^4}\, d\eta = 0. \tag{2.27}$$

From the integral for G_2 given in Appendix A it follows that the left-hand side of (2.27) is equal to $2^{\frac{7}{4}}\pi^{-\frac{3}{2}}(-\frac{1}{4}!)^3\alpha_1 B_1/5$, and so either $\alpha_1 = 0$ or $B_1 = 0$.

When a similar difficulty occurred in the incompressible theory, it was not necessary to infer that the coefficient of the integral must vanish, for by adding a suitable term with a factor $\log\xi$ the difficulty was overcome. The reason is that it was possible to choose the stage ($n = 5$) at which the new term was added, so that no additional integral condition need be satisfied. In the present instance, however, this is not possible. The only suitable place might be as a modification to f_1, but since the equations for f_2 depend on f_1^2, essentially there must be a term with a factor $(\log\xi)^2$ added to f_2. Although no integral condition is needed to make this term acceptable, there are integral conditions associated with the term having a factor $\log\xi$ and the term independent of $\log\xi$. Accordingly, the arbitrary constant in the modification to f_1 must satisfy two conditions and is either zero or over-specified.

3. Non-zero heat transfer at the origin

The two alternatives presented by the condition $\alpha_1 B_1 = 0$, obtained at the end of the last section, are reduced to simply

$$\alpha_1 = 0 \tag{3.1}$$

if the heat transfer does not vanish at the origin where the skin-friction vanishes. In this section we show that if $B \neq 0$ the solution must be completely regular near the origin. The method is to show that at each stage of the expansion only regular terms are introduced into the solution. We have from the last section

$$f_1 = \tfrac{1}{24}B_1\eta^4, \quad g_1 = B_1\eta, \quad f_2 = \alpha_2\eta^2, \quad g_2 = 0, \tag{3.2}$$

where, on substituting into (2.13) and (2.14), all terms except f_2 contribute only positive integral powers of X, Y to ψ. It is now shown that $\alpha_2 = 0$. For the equation for g_3 is

$$g_3'' - \tfrac{1}{2}\eta^3 g_3' + \tfrac{3}{2}\eta^2 g_3 = 5f_2\, g_1' - f_2' g_1 = 3\alpha_2 B_1 \eta^2, \tag{3.3}$$

with solution

$$g_3 = 2\alpha_2 B_1\{1 - G_3(\eta)\}. \tag{3.4}$$

Substitution into (2.11) yields, as the equation for f_3,

$$f_3''' - \tfrac{1}{2}\eta^3 f_3'' + \tfrac{7}{2}\eta^2 f_3' - 6\eta f_3 = \tfrac{1}{2}\alpha_2 B_1\eta^4 + 2\alpha_2 B_1\{1 - G_3(\eta)\}. \tag{3.5}$$

Since the forcing term of (3.5) is arbitrary to the extent of a multiplicative factor only, we can expect f_3 to be algebraic at infinity and to have a double zero at the origin, if and only if $\alpha_2 = 0$. In fact this has been verified numerically, using the method described in Appendix B. Hence

$$g_3 = 0, \quad f_3 = \alpha_3\eta^2, \tag{3.6}$$

where α_3 is arbitrary. The new term, when expressed in terms of X and Y, is regular at the origin contributing nothing to S and a term of order XY^2 to ψ.

The equation for g_4 is

$$g_4'' - \tfrac{1}{2}\eta^3 g_4 + 2\eta^2 g_4 = 4\alpha_3 B_1 \eta^2, \tag{3.7}$$

the boundary condition at the wall being now $g_4(0) = S_1$, so that the solution required is

$$g_4 = \tfrac{1}{3}\alpha_3 B_1 \eta^4 + S_1(1 - \tfrac{1}{6}\eta^4), \tag{3.8}$$

which again corresponds to a regular solution at the origin. Now the purpose of this section is to show that if $B_1 \neq 0$ the solution must be regular at the origin. In order to keep the argument as simple as possible we shall set $S_1 = 0$, although there is no formal difficulty in the more general problem. The equation for f_4 is then

$$\begin{aligned} f_4''' - \tfrac{1}{2}\eta^3 f_4'' + 4\eta^2 f_4' - 7\eta f_4 &= g_4 + P_1 + 6f_1'' f_3 + 4f_3'' f_1 - 8f_1' f_3' \\ &= P_1 + \alpha_3 B_1 \eta^4, \end{aligned} \tag{3.9}$$

with solution

$$f_4 = \alpha_4 \eta^2 + \tfrac{1}{6}P_1(\eta^3 - \tfrac{1}{105}\eta^7) + \tfrac{1}{210}\eta^7 \alpha_3 B_1. \tag{3.10}$$

Of these terms, only $\alpha_4 \eta^2$ corresponds to a singular solution at the origin. The equation for g_5 is

$$\begin{aligned} g_5'' - \tfrac{1}{2}\eta^3 g_5' + \tfrac{5}{2}\eta^2 g_5 &= 7f_4 g_1' - f_4' g_1 + 4f_1 g_4' - 4f_1' g_4 \\ &= 5\alpha_4 B_1 \eta^2 + \tfrac{2}{3}P_1 B_1 \eta^3. \end{aligned} \tag{3.11}$$

Since the complementary function of (3.11), algebraic at infinity, is given by $\eta - \frac{1}{10}\eta^5$ which vanishes at the origin, the appropriate solution of (3.11) is found by introducing a factor $\log \xi$ into g_5. We write

$$g_5 = 5\alpha_4 B_1[C_5(\eta - \tfrac{1}{10}\eta^5)\log \xi + h_5(\eta)] + \tfrac{1}{30}P_1 B_1 \eta^5, \tag{3.12}$$

the equation satisfied by h_5 being

$$h_5'' - \tfrac{1}{2}\eta^3 h_5' + \tfrac{5}{2}\eta^2 h_5 = \eta^2 - \tfrac{1}{2}C_5 \eta^3(1 - \tfrac{1}{10}\eta^4). \tag{3.13}$$

In the usual way it can be shown that h_5 vanishes at the origin and is algebraic at infinity only if

$$\int_0^\infty e^{-\frac{1}{8}\eta^4} \eta^3 \left(1 - \frac{\eta^4}{10}\right)\left\{1 - \frac{\eta}{2}C_5\left(1 - \frac{\eta^4}{10}\right)\right\} d\eta = 0, \quad \text{i.e. if} \quad 2^{-\frac{1}{4}}(\tfrac{1}{4})!\,C_5 = 1. \tag{3.14}$$

From the form of $g_5(\eta)$ given in (3.12) it follows that $f_5(\eta)$ must include a factor proportional to $\log \xi$ and so we write

$$f_5(\eta) = p_5(\eta)\log \xi + q_5(\eta), \tag{3.15}$$

p_5 and q_5 having double zeros at the origin and being algebraic at infinity. The equation for p_5 is

$$p_5''' - \tfrac{1}{2}\eta^3 p_5'' + \tfrac{9}{2}\eta^2 p_5' - 8\eta p_5 = 5\alpha_4 B_1 C_5(\eta - \tfrac{1}{10}\eta^5), \tag{3.16}$$

with solution

$$p_5 = \beta_5 \eta^2 + \tfrac{5}{24}\alpha_4 B_1 C_5(\eta^4 - \tfrac{2}{105}\eta^8), \tag{3.17}$$

where β_5 is arbitrary. The equation for $q_5(\eta)$ is

$$\begin{aligned} q_5''' - \tfrac{1}{2}\eta^3 q_5'' + \tfrac{9}{2}\eta^2 q_5' - 8\eta q_5 &= (\eta p_5 - \tfrac{1}{2}\eta^2 p_5') + 5\alpha_4 B_1 h_5 + \tfrac{1}{30}P_1 B_1 \eta^5 + P_1 g_1 \\ &\quad + 7f_1'' f_4 + 4f_4'' f_1 - 9f_1' f_4' \\ &= P_1 B_1(\eta + \tfrac{1}{30}\eta^5) + 5\alpha_4 B_1(h_5 - \tfrac{1}{6}\eta^4 - \tfrac{1}{12}\eta^5 C_5 + \tfrac{2}{35}\eta^9 C_5). \end{aligned} \tag{3.18}$$

The term containing $P_1 B_1$ only contributes a polynomial of degree eight to q_5, so that the terms which are exponentially large at infinity can only arise from the second forcing term of (3.18). This term is fully determinate apart from the multiplicative constant and accordingly we can expect the contribution it makes to q_5 to be exponentially large at infinity, except when $\alpha_4 = 0$. This has been verified by actual integration after expressing h_5 as an infinite series and expressing the condition that q_5 be not exponentially large at infinity in a similar integral form to (2.26).

With $\alpha_4 = 0$ all the terms of the expansion which have been worked out so far are, when expressed in terms of X, Y, regular at separation, apart from $\beta_5 \eta^2$. This terms occurs in the expression for $p_5(\eta)$ and now that $\alpha_4 = 0$ there is no obvious reason why it should be retained. If nevertheless it is left in and the corresponding equation for p_6 examined, it follows that $\beta_5 = 0$ in a similar way to $\alpha_1 = 0$.

It is generally true in fact that if the expansion of f as far as $\xi^{n-1} f_{n-1}$ is regular when expressed in terms of X, Y the contribution from f_n will also be regular apart from a term $\alpha_n \eta^2$ $(n \neq 4r+3)$. The equation for f_{n+1} then leads to a solution which is exponentially large at infinity unless the product of $\alpha_n B_1$ and a certain fully determinate integral vanishes. In the three typical cases considered above, this integral has been shown to be non-zero. Since there is no *a priori* reason why the integral should vanish from the properties of the functions involved, one might reasonably assume that it does not, although a numerical integration or summation would be needed in any specific case. Hence we conclude that, if $B_1 \neq 0$, the solution must be regular at separation.

If $n = 4r+3$, however, f_{n+1} is not exponential at infinity even if $\alpha_n \neq 0$ and indeed the contribution to ψ from f_n is regular in terms of X, Y. Accordingly, the expansion of ψ contains an infinite number of arbitrary constants α_{4r+3} even though it is regular at the origin. Further B_{4r+1}, being the coefficients of those polynomial solutions of $g_n = 0$ which vanish at $\eta = 0$, are also arbitrary and also make only regular contributions to ψ, S.

We conclude that, if the heat transfer at the point of vanishing skin-friction is non-zero in this model boundary layer, then the solution is regular at this point. Further if ψ, S are expanded in a double power series in X, Y, none of the terms in $(\partial^2\psi/\partial Y^2)_{Y=0}$, $(\partial S/\partial Y)_{Y=0}$, i.e. in the expansion of the skin friction and the heat transfer, are determined solely by local conditions; they depend in some way on the prescribed behaviour of ψ, S at the initial station of X.

4. Zero heat transfer at the point where the skin-friction vanishes

In this section we consider the alternative condition for there to be a solution of the boundary-layer equations near the origin of the form assumed, namely that the heat transfer should be zero at the origin though it need not vanish anywhere else. We shall show that the solution is singular just as in the incompressible case. For, if $B_1 = 0$ from (2.24), $g_2 = 0$ and continuing the argument $g_3 = 0$ too, while f_1, f_2, f_3 are exactly as given by Goldstein (1948). This is true whether S is prescribed at the wall or the heat transfer is zero. They could only be non-zero if the prescribed wall temperature were not regular at separation, which is excluded. The equation for g_4 is now

$$g_4'' - \tfrac{1}{2}\eta^3 g_4' + 2\eta^2 g_4 = 0$$

and $g_4(0) = S_1$ so that, from Appendix A,

$$g_4(\eta) = S_1(1-\tfrac{1}{6}\eta^4). \tag{4.1}$$

The equation for $g_5(\eta)$ is

$$g_5'' - \tfrac{1}{2}\eta^3 g_5' + \tfrac{5}{2}\eta^2 g_5 = -8\alpha_1 S_1\eta(1+\tfrac{1}{6}\eta^4) \tag{4.2}$$

and $g_5(0) = 0$. The appropriate solution is

$$g_5(\eta) = -\tfrac{4}{3}\alpha_1 S_1\eta^3 + B_5(\eta - \tfrac{1}{10}\eta^5). \tag{4.3}$$

Notice that one might have expected the solution of (4.3) to lead to difficulties since all complementary functions which do not vanish at the origin are exponentially large at infinity. Hence, if the particular integral were exponentially large at infinity, it could not be cancelled by a complementary function without making $g_5(0) \neq 0$. It appears that the particular integral is a polynomial so that the difficulty does not arise. It must not be thought that the simplicity of (4.3) is fortunate and essential to the argument; indeed if more general boundary layers are considered, in which for example, the Prandtl number is not unity, the simplicity is lost. However, in that case it is only necessary to add to $g_5(\eta)$ an extra term

$$D_5\log\xi(\eta - \tfrac{1}{10}\eta^5), \tag{4.4}$$

where D_5 is a constant chosen to cancel this exponential term as outlined in § 3 above and in Stewartson (1958). Further, such a term would alter the equation for $f_5(\eta)$ but again could be dealt with by adding to $f_4(\eta)$ a function of η multiplied by at worst $\log\xi$; such a term would not affect $g_5(\eta)$. Accordingly, there is no need to check from the equations for $g_{4r+1}(\eta)$, r integral, that the particular integral is algebraic at infinity: it does not matter.

From now on the procedure is iterative. The equation for g_6,

$$g_6''' - \tfrac{1}{2}\eta^3 g_6' + 3\eta^2 g_6' = 4f_1 g_5' - 5f_1' g_5 + 5f_2 g_4' - 4f_2' g_4, \tag{4.5}$$

determines $g_6(\eta)$. The corresponding equation for $f_6(\eta)$ presents no additional difficulties to the incompressible problem (Stewartson 1958). Again it is necessary to modify $f_5(\eta)$ by adding a term $\beta_5\eta^2\log\xi$ to it, to modify f_6 in consequence, and it is then possible to determine β_5 in terms of B_5 while α_5 remains arbitrary. Continuing the argument, it follows that if $B_1 = 0$ a solution which is singular at the origin can be found containing two infinite sets of arbitrary constants. One of these is α_{4r+1} $(r = 0, 1, 2, \ldots)$. If the wall temperature is prescribed the other is B_{4r+1} $(r = 1, 2, \ldots)$, while if the heat transfer from the wall is zero the other is S_n $(n = 0, 1, 2, \ldots)$.

5. Generalization of the theory

For a real fluid, the appropriate equations for a compressible boundary layer are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{5.1}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{u}{c_p}\frac{dP}{dx} = \frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial T}{\partial y}\right) + \frac{\mu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{5.2}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{5.3}$$

where u, v are the components of velocity along and perpendicular to the wall, x, y denote distances measured along and perpendicular to the wall, P is the pressure in the main stream just outside the boundary layer, ρ is the density, μ is the viscosity, T is the temperature, σ is the Prandtl number and c_p the specific heat of the gas at constant pressure. The correspondence between (5.1) to (5.3) and (2.1), (2.2) may be found in Stewartson (1949). The boundary conditions are of the same form as those given earlier, viz. u, T tend to prescribed values as $y \to \infty$ (corresponding to (2.3)), $u = v = 0$ at $y = 0$ (corresponding to (2.4)), and T has a prescribed behaviour at $y = 0$ (corresponding to (2.3), (2.6)).

In considering the form of the solution near a point of zero skin-friction, the crux of the matter is whether the leading term in the temperature is or is not a constant. This question occurred in the discussion of the model fluid and of equations (2.18) with solution (2.19). A similar set of equations is obtained from (5.1) to (5.3), but of a slightly more complex nature because μ is a function of temperature. It turns out that, for exactly the same reason as before, the solution of this pair contradicts the assumptions on which the theory is built, namely that the stress first vanishes at the wall and that the singularity is centred at the point of zero skin-friction. We shall not give the argument here, for although straightforward it is complicated. The essential point is that so far as the leading term is concerned $(u/c_p)\, dP/dx$ and $(\mu/c_p)\,(\partial u/\partial y)^2$ may be neglected in (5.2) whence it takes on the same form as (2.2). Accordingly, the corresponding equation to that for g_0 in (2.18) is similar and has the same properties.

Once we have established that the leading term in the temperature is constant for both the model and the real fluids we can see that two of the complications introduced by the real fluid are perturbations and hence do not affect the character of the solutions obtained earlier. Thus the variation of viscosity with temperature is a perturbation and, so long as the solution is regular, will only make regular contributions to the successive equations. The critical integrals which must vanish if the singular solution is to be acceptable are therefore still dependent only on $\alpha_n B_1$, using the notation of the previous sections, and in fact as before will be non-zero so that the solution must be regular. Again, the dissipation term in (5.2) may be removed by means of a particular integral if $\sigma = 1$, but in any case it is easily seen to be of the order of $\xi^4 T$, so that its effect is noticed for the first time in the equation for $g_4(\eta)$ only. For the same reason as for μ, this means that it will have no effect on the critical integrals.

There remains only the effect of Prandtl number σ. So far as the critical features of the solution are concerned, its effect is strictly equivalent to replacing ν_0 by ν_0/σ in (2.2) while leaving (2.1) alone. Accordingly, the main change in the equations of the expansion is to replace $g_n''(\eta)$ at each stage by g_n''/σ. Thus the equation for g_2 changes from (2.23) to

$$\sigma^{-1} g_2'' - \tfrac{1}{2}\eta^3 g_2' + \eta^2 g_2 = 2\alpha_1 B_1 \eta^2, \tag{5.4}$$

so that its solution also involves σ. The critical integral associated with g_2, and in the same way with all other g_n $(n \neq 4r+4)$, will therefore also contain σ, which makes for greater complication. It has unfortunately not been possible to prove much about the effect of σ on the critical integrals, but from an examination of a few special cases there is no indication that the integrals vanish for any σ.

Finally, it is noted that the effect of suction or injection may be taken into account. Just as Terrill (1960) found in the incompressible boundary layer, its effect is of degree rather than character and it does not affect the nature of the singularity.

We conclude therefore that for a general compressible boundary layer of a real fluid the flow in the neighbourhood of the point of zero skin-friction is only singular if the heat transfer at the wall vanishes there. It is noted that an attempt was made to see if a singularity could occur at the wall at a point where the skin friction is not zero using Goldstein's method (1930), but the conclusion was that it could not. Since the difficulty in the numerical integration mentioned by Curle (1958) clearly indicates a singularity somewhere, we must conclude that it is centred in the interior of the fluid; the behaviour of the solution near such a singularity is, however, beyond the scope of this paper.

Appendix A

The homogeneous differential equation satisfied by the temperature function g_n of § 2 is

$$g_n'' - \tfrac{1}{2}\eta^3 g_n' + \tfrac{1}{2}n\,\eta^2 g_n = 0. \tag{A.1}$$

This equation may be solved by the methods of contour integration by writing

$$g_n = \int_c e^{-\frac{1}{8}\eta^4 x} p(x)\,dx,$$

from which it may be shown that

$$g_n = \int_c e^{-\frac{1}{8}\eta^4 x} \frac{(x+1)^{\frac{1}{4}(n-1)}}{x^{\frac{1}{4}n}} \frac{dx}{x}, \tag{A.2}$$

the basic contours starting at $x = -1$, ending at $x = \infty \pm 0i$ and passing either above or below the singularity at $x = 0$. It is clear from (A. 2) that g_n is a polynomial only if $n = 4r$ or $4r+1$ $(r = 0, 1, 2, \ldots)$. If $n = 4r$ the polynomial solution is non-zero at $\eta = 0$ but its derivative is zero at $\eta = 0$, whereas the reverse is true if $n = 4r+1$. Of particular importance is that solution $G_n(\eta)$, if it exists, such that $G_n(0) = 1$ and G_n is algebraic at infinity. This is given by

$$G_n(\eta) = A_n \mathscr{F} \int_0^\infty e^{-\frac{1}{8}\eta^4 x} \frac{(x+1)^{\frac{1}{4}(n-1)}}{x^{\frac{1}{4}n}} \frac{dx}{x} \quad (n \neq 4r), \tag{A.3}$$

where the symbol $\mathscr{F}$ is used to denote the finite part of the infinite integral and

$$\frac{(-\frac{3}{4})!\,(-\frac{1}{4}n-1)!}{(-\frac{1}{4}n-\frac{3}{4})!} A_n = 1. \tag{A.4}$$

If $n = 4r$, (A. 3) is not directly of use since $A_n = 0$, but then the polynomial solution can easily be obtained from (A. 2).

Appendix B

On writing

$$f_3 = \tfrac{1}{3}\alpha_2 B_1 \eta^3 - 2\alpha_2 B_1 F(\eta),$$

equation (3.5) reduces to

$$F'''' - \tfrac{1}{2}\eta^3 F''' + \tfrac{7}{2}\eta^2 F'' - 6\eta F = G_3(\eta). \tag{B.1}$$

The boundary conditions are $F(0) = F'(0) = 0$, and we wish to show that F must be exponential at infinity. One complementary function of (B. 1) is η^2, and if that one of the other two which is algebraic at infinity is denoted by $L(\eta)$, it follows from the known properties of (B. 1) (Terrill 1960, p. 62) that

$$L(\eta) = 2^{\frac{1}{4}} \sum_{m=0}^{\infty} \frac{(m-\frac{9}{4})!\,\eta^{4m+1}}{(m+\frac{1}{4})!\,m!\,(4m-1)\,8^m} - \sum_{m=0}^{\infty} \frac{(m-\frac{10}{4})!\,\eta^{4m}}{(m-\frac{1}{4})!\,m!\,(2m-1)\,8^m}. \tag{B.2}$$

Further, on solving (B. 1) by the method of variation of parameters, it follows that F is exponential at infinity if

$$\int_0^\infty e^{-\frac{1}{8}\eta^4}(\eta^2 L' - 2\eta L)\, G_3(\eta)\, d\eta \neq 0. \tag{B.3}$$

Suppose now that L contains a term η^s. Its contribution to the integral in (B. 3) is

$$\begin{aligned}(s-2)\int_0^\infty e^{-\frac{1}{8}\eta^4}\eta^{s+1} G_3(\eta)\, dy &= \frac{(s-2)\,A_3}{2^{\frac{1}{2}}}\,\mathscr{F}\int_0^\infty \frac{(x+1)^{-\frac{1}{4}s}}{x^{\frac{7}{4}}}\, 8^{\frac{1}{4}s}(\tfrac{1}{4}s-\tfrac{1}{2})!\, dx \\ &= (s-2)\,A_3\, 8^{\frac{1}{4}s}(\tfrac{1}{4}s-\tfrac{1}{2})!\,(\tfrac{1}{4}s-\tfrac{1}{4})!\,(-\tfrac{7}{4})!\,[2^{\frac{1}{2}}(\tfrac{1}{4}s-1)!]^{-1},\end{aligned} \tag{B.4}$$

using (A. 3). Hence, substituting (B. 2) into (B. 3) and using (B. 4), we find that the integral in (B. 3) is equal to

$$2^{\frac{1}{2}} A_3(-\tfrac{7}{4})! \sum_{m=0}^{\infty}\left\{\frac{(m-\frac{9}{4})!\,(m-\frac{1}{4})!}{(m+\frac{1}{4})!\,(m-\frac{3}{4})!} - \frac{(m-\frac{5}{2})!\,(m-\frac{1}{2})!}{(m-1)!\,m!}\right\}. \tag{B.5}$$

The second series can be evaluated from the theory of hypergeometric series and is equal to $-\frac{4}{3}$. The first series has been summed numerically and is not equal to $-\frac{4}{3}$, thus showing that f_3 is exponentially large at infinity unless $\alpha_2 = 0$.

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